The Empty Number Line

Associate Professor Janette Bobis
Mathematics Education, Faculty of Education and Social Work
Associate Dean, Division of Research
University of Sydney

Many of us will have been amazed at how some people have this seemingly innate ability to quickly calculate answers in their heads to the most complex mathematical computations, while so many others seem to struggle with much simpler questions, even with the assistance of pencil and paper or a calculator.

My name is Janette Bobis and I teach mathematics education to prospective primary teachers in the Faculty of Education and Social Work at the University of Sydney. I want to talk about the empty number line, a relatively new and powerful instructional tool that we have to assist students to develop more efficient mental computational strategies.

Mental computation is now receiving greater emphasis in the curriculum and in classrooms across the world than ever before. We've known for quite some time about the importance of being able to calculate mentally and recognised that 'good' mental calculators in the past were quite flexible in their strategy use and generally used their own 'invented' mental strategies — those that were not taught at school or conformed to any traditional, standard, pencil and paper method.

Such 'intuitive' strategies are potentially quite powerful because they make sense to the user. While not all invented strategies are 'efficient' in all situations, for example, counting-on by ones from the larger number to determine the answer to 34 + 3, it might be efficient; but using the same strategy to calculate 34 + 26, is not. It's our role as teachers to guide students who would not ordinarily develop more sophisticated strategies by themselves to progress to using more efficient ones. Fortunately, educational researchers have developed and studied the impact of instructional tools to assist us in the process of teaching and learning mental computational strategies. One such tool is the empty number line, sometimes referred to as the blank number line.

So, what exactly is an empty number line? And, how is it different from the traditional ‘structured’ number line that contains sequences of numbers and marks, and usually involving counting by ones?

Let’s start by examining the structured number line first, as this is one most of us are more familiar with.

The structured number line is a ‘measurement model’. By this we mean that numbers are representations of lengths, rather than simply points on a line that they are labelling. The proximity or distance of an ‘unknown’ number to a known number is important because this enables a student to determine the unknown. The lengths or proportion of the line between the numbers and marks provide clues as to the value of a missing number.

On the other hand, the empty number line is a ‘counting model’. The marks on the empty number line are not meant to be proportional. This means that the lengths or distances between the marks will not indicate their value. That’s because the empty number line’s purpose is very different from that of the structured number line.

The empty number line is a visual representation for recording and sharing students’ thinking strategies during the actual process of mental computation.

Starting with an empty line — a number line with no numbers or markers — students only mark the numbers they need for their calculation. For example, if using an empty number line to assist solving 58 + 24, a student would start with 58, jump ahead 20 to 78, then jump ahead 4 more using either ones or a jump of 2, then two more to reach 82. You’ll note that zero, or any other unnecessary number, is not represented on the empty number line. Redundant numbers will only interfere with the process.

The structured number line and the empty number line are fundamentally different because they are used for different purposes. They therefore require students to possess different skills and knowledge to effectively use them, so we can't assume that proficiency in working with one type of number line will automatically transfer to the other.

To truly appreciate the value of the empty number line and understand how best to teach it, it is helpful for teachers to know about its development and what works and what doesn't work when it is introduced to young children.

The empty number line was originally proposed as a central model for addition and subtraction by researchers from the Netherlands in the 1980s. It still holds a prominent position in the Netherlands' curriculum and is prevalent in regular classroom instruction throughout that country. Since then, the use of empty number lines has been advocated by curriculum documents in the UK, New Zealand and New South Wales.

Importantly, it was originally developed out of a need for a new tool to help overcome problems experienced by children when performing addition and subtraction involving two-digit numbers. Such problems included the common 'procedure-only' use of base 10 materials (such as Multi-Attribute Blocks or MAB) when modelling the computational procedures — particularly for subtraction when regrouping was involved and the standard written algorithms were being used. For example, a common error made when solving 53 – 26 using either base 10 materials or a standard pencil and paper algorithm, is for a child to calculate the digits in the ones column first and, knowing from an early age that they must take the little number away from the larger one, children commonly reverse the order and take 3 from 6.

When the empty number line is used to support a solution strategy, this error is unlikely to be made, since the calculation proceeds in a linear and sequential fashion proceeding left to right when the computation is written in a horizontal format. So starting at 53 on the empty number line, the child would jump back 20 (or two lots of 10) to 33, and then jump back another 6, possibly using two lots of 3 jumps to arrive at 27.

In this way, the empty number line reinforces the use of a very powerful mental strategy for calculations involving two digits or more. The jump strategy, or sequential method that is supported by the empty number line model, treats the first number as a whole. It is not split into tens and ones as is required when traditional pencil and paper procedures are implemented. The second number can then be partitioned in any number of convenient ways to assist addition or subtraction. For instance, in the case of 48 + 25, a student might choose to jump from 48 to 58, then 68 using two lots of tens, then add 5 more using five jumps of one.

A student with more experience using the strategy and a more advanced understanding of number relationships might start at 48, make one jump of 20 to 68, then partition the 5 into a 2 and a 3. In this way, the student could conveniently build the 68 to the next 10 — being 70 — and then finally add the 3 to make 73. There are other sequences of jumps that could also be used also to derive the correct answer, but the important thing is that students choose combinations that make sense to them.

As teachers, it is important that we monitor student progress to ensure that their use of the strategy becomes progressively more sophisticated. The empty number line is a representational tool that not only scaffolds students' thinking along the path to a more abstract level, but also allows that thinking to become visible. It scaffolds their thinking because it allows them to see what parts have been calculated and what parts remain, thus reducing the cognitive load for a young child who is still learning how to master mental thinking strategies. With a visible record of their strategy use, teachers then have the knowledge to match instruction to each child's level of development.

Further advantages of using the empty number line include the need for a linear representation of number. We've already seen how the empty number line encourages thought processes that are linear or sequential. That is, treating the first number as a whole and working from left to right when operating on the second number. This reduces the amount of information a child is required to keep in working memory if they have to split both numbers up into tens and ones, add the digits separately, carry over any additional tens created by operating on the ones, and then join the tens and ones back together again at the end.

This 'split' method can be effective for mental calculations where there is no need to carry over tens, such as when adding 63 and, say, 22. However, it is extremely cumbersome when the numbers get larger and when carrying tens is involved. Research has shown that the success rate of students who solely rely on the split strategy for mental computation is far less than those who are competent users of the jump strategy.

Another benefit of the empty number line is that it reflects more closely the intuitive mental strategies already used by young children. For instance, children naturally tend to focus first on counting strategies to solve number problems up to 100 — 'counting-on' or 'counting-down'.

More proficient mental calculators use a combination of counting strategies, usually in chunks of 10, with partitioning strategies. 'Partitioning' involves children 'taking apart' numbers in flexible ways to make them more convenient to calculate mentally. These strategies normally approximate the jumps on a number line.

Importantly, any instructional tool does not stand alone; it must be accompanied by appropriate instruction. Ideally, at some point, the student must be able to mentally compute without the assistance of the empty number line or any other visible support mechanism. So how does instruction proceed so that students can eventually work at an efficient and abstract level of mental computation?

Instructional sequences must consider the prior experiences and knowledge of the students it is being introduced to, no matter what their chronological ages. An important precondition to allow exploration of addition and subtraction, is a thorough understanding of numbers to 100. Also, while the structured number line and the empty number line are fundamentally different models and serve different purposes, there are some prerequisite skills and knowledge that can be derived from working with the structured number line that will assist students' understanding and use of the empty number line.

Students obviously need counting skills. First, counting by ones, but not always starting from 1. They must be able to count forwards and backwards starting from any number.

At first, a structured number line can be used to support this process. For example, using a number line with numbers 1 to 10 or 1 to 20, students can place their finger or a peg on a start number, such as 8 and then count forwards or back by any number.

This skill can then be developed to counting by tens, both on and off the decade, forwards and backwards. Again a structured number line might be useful or a 'hundreds chart' to scaffold the process.

As well as being able to count in sequence, another important skill for working on number lines, particularly structured number lines, is the ability to locate a number in relation to other numbers. Partial number lines are also useful for developing this skill and make a useful transition to the empty number line.

Importantly, though, the partial number line is also a measurement model as it relies on proportion and the length between marks on the line to determine a number's value. So, if given a partial number line with only the numbers 0 and 100 marked on it, a student would use proportional reasoning to determine where '50' might lie on the line.

Two other essential strategies that children must understand and use effectively before a more sophisticated use of the empty number line is possible include: making ten; and jumping across tens, or bridging tens.

Making ten requires that children know their basic number combinations to 10.

Bridging tens requires that children are able to flexibly partition numbers. For example, to solve 8 + 5, the first number remains as a whole and the 5 is partitioned and added in parts. It makes it easier, if a part of the 5 is added to the 8 to 'make 10' before the final part is added. So: 8 + 2 = 10; and then 10 + 3 = 13.

This same strategy can then be applied when bridging tens in higher decades. For example, to solve 38 + 5: 38 + 2 = 40; and then the 40 + 3 = 43. These steps can be demonstrated on a structured number line or an empty number line and recorded symbolically as we've seen here.

Based on the strategies for counting in tens, making tens, bridging tens, and having a familiarity with structured and partially structured number lines, students can then make the transition to the empty number line and the jump strategy for two-digit addition and subtraction quite smoothly.

For calculations such as 66 – 29, it is often more efficient to apply a compensation strategy, such as subtracting 30 and adding 1. For instructional purposes, it's important that only the numbers needed be recorded on the empty number line and that the number of jumps decreases with increased sophistication of the strategy's use.

Teachers should encourage verbalisation and the symbolic recording of solutions that students have constructed for themselves. In this way, you will be encouraging the progression to more abstract levels of thinking. Remember, the ultimate goal is for students to effectively calculate mentally without the need for external support models. The expected advantages of using the empty number line may be lost if we unnecessarily prolong its use as a visual modelling strategy, particularly when children have progressed beyond it.

Other ways in which we can unwittingly use the empty number line so that it may have a detrimental impact on student strategy development is if we: over-regulate the use of the empty number line; provide pre-drawn lines with numbers that we anticipate students may need; or prescribe the use of the empty number line when the jump strategy is not the most appropriate one.

Requiring students to always draw a neatly represented visual model can become mundane and contributes little or anything to the effectiveness of its implementation. Prepared lines with even a few numbers on them can confuse students or even encourage them to use a lower-level strategy than they would ordinarily use, particularly if numbers appear in a counting-by-ones sequence when we are really trying to encourage non-count-by-ones strategies when it comes to operations involving two-digit numbers.

Recent investigations by Marja van den Heuvel-Panhuizen in the Netherlands, refer to similar concerns about the introduction and appropriate application of the empty number line.

Finally, we can't forget about the child's perspective in all of this. For adults who already 'know' the mathematics, the empty number line can seem like a straightforward, convenient instructional tool to assist children with mental calculation. But for children who don't have the same conceptual understanding, it's still a challenge to learn to use it correctly. And for some children, this will take time and appropriate guidance. The ability to flexibly use mental computation strategies is a high priority in our schools and in our society. The empty number line is one tool that can help students achieve this goal.